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New uncertainty relations for tomographic entropy: application to squeezed states and solitons

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Abstract. Using the tomographic probability distribution (symplectic tomogram) describing the quantum state (instead of the wave function or density matrix) and properties of recently introduced tomographic entropy associated with the probability distribution, the new uncertainty relation for the tomographic entropy is obtained. Examples of the entropic uncertainty relation for squeezed states and solitons of the Bose-Einstein condensate are considered.

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1 Introduction

Quantum mechanics is known to differ from classical mechanics due to the existence of the position-momentum uncertainty relation by Heisenberg [1,2]. The uncertainty relation containing the correlation of the position and momentum was found by Robertson [3] and Schrödinger [4–6]. There exists the uncertainty relation of the position and momentum for mixed quantum states [7]. New kinds of the uncertainty relations were obtained by Trifonov [8]. Extensions of the uncertainty relations of [9] for mixed states were found by Karelin [10]. Review of the uncertainty relations in quantum mechanics is given in [11,12].

There exist specific uncertainty relations called "entropic uncertainty relations" based on the notion of Shannon entropy and information [13]. These relations, which read as inequalities for entropy associated with the position-and-momentum probability distributions, were discussed, for example, in [14,15].

Recently a new formulation of quantum mechanics where the quantum states are described by tomographic-probability distributions (instead of the wave function or density matrices) was suggested [16]. For a system with continuous degrees of freedom, such probability is the symplectic tomogram of the quantum state [17]. The corresponding symplectic tomographic entropy was introduced for quantum states in [18] and in signal analysis in [19]. In [20] the symplectic entropy was discussed for the BEC solitons, in view of the tomogram of the solution to Gross-Pitaevskii equation. A general approach to quantum information including the application of different kinds of tomographic entropies was developed in [21].

The aim of this study is to establish a new kind of entropic uncertainty relations formulated as inequality for the entropy associated with the symplectic tomogram of the quantum state of a system with continuous degrees of freedom.

The paper is organized as follows.

In Section 2, a review of known entropic uncertainty relations for systems with continuous variables is presented while, in Section 3, the symplectic-tomography approach is discussed. Entropic inequalities for symplectic entropy are studied in Section 4 and examples of new inequalities for the Gaussian packets (squeezed states) and soliton solution of Gross-Pitaevskii equation (Bose-Einstein condensates) are given in Section 5. Finally, conclusions are summarized in Section 6.

2 Entropy and entropic uncertainty relations

In the context of information theory, entropy is related to an arbitrary probability-distribution function [13]. For example, given the probability distribution $P(n)$, where n

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is a discrete random variable, i.e.,

$$P(n) \geq 0, \tag{1}$$

together with the normalization condition

$$\sum_n P(n) = 1, \tag{2}$$

one has, by definition, the entropy

$$S = -\sum_n P(n) \ln P(n) = -\langle \ln P(n) \rangle. \tag{3}$$

In quantum mechanics, the discrete probability distributions are standard ingredients in the description of spin (see, e.g., tomographic probability of spin states in [22,23] and related entropy for spin tomograms in [24]).

For continuous variables, the wave function $\psi(x)$ provides the probability-distribution density

$$P(x) = |\psi(x)|^2. \tag{4}$$

The corresponding entropy reads (see, e.g., [7])

$$S_x = -\int |\psi(x)|^2 \ln |\psi(x)|^2 \, dx. \tag{5}$$

In the momentum representation, one has the wave function

$$\widetilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int \psi(x) e^{-ipx} \, dx \qquad (\hbar = 1). \tag{6}$$

The corresponding entropy related to the momentum-probability density $|\widetilde{\psi}(p)|^2$ reads

$$S_p = -\int |\widetilde{\psi}(p)|^2 \ln |\widetilde{\psi}(p)|^2 \, dp. \tag{7}$$

It is worthy noting that one can construct entropies S_x and S_p not only in quantum mechanics. If the function $\psi(x)$ is replaced by a signal function $f(t)$ depending on time t, the function $\widetilde{\psi}(p)$ is replaced by the function $\widetilde{f}(\omega)$ describing the signal spectrum.

In this case, the entropy of the signal

$$S_t = -\int |f(t)|^2 \ln |f(t)|^2 \, dt \tag{8}$$

and its spectrum

$$S_\omega = -\int |\widetilde{f}(\omega)|^2 \ln |\widetilde{f}(\omega)|^2 \, d\omega \tag{9}$$

provide some information characteristics of the signal.

From mathematical point of view, there exists the correlation of entropies S_x and S_p (S_t and S_ω), since the function $\psi(x)$ [$f(t)$] determines the Fourier component $\widetilde{\psi}(p)$ [$\widetilde{f}(\omega)$]. This means that the entropies S_x and S_p have to obey some constrains. These constrains are entropic uncertainty relations (some inequalities).

For the one-mode system, the inequality reads (see [7])

$$S_x + S_p \geq \ln(\pi e), \tag{10}$$

or

$$S_t + S_\omega \geq \ln(\pi e). \tag{11}$$

For the Gaussian wave functions (Gaussian signals) describing the states without correlations of the position and momentum, e.g., the ground state of the harmonic oscillator

$$\psi(x) = \pi^{-1/4} e^{-x^2/2}, \qquad \widetilde{\psi}(p) = \pi^{-1/4} e^{-p^2/2}, \tag{12}$$

one has

$$S_x^{(0)} = S_p^{(0)} = \frac{1}{2} \ln(\pi e). \tag{13}$$

Consequently,

$$S_x^{(0)} + S_p^{(0)} = \ln(\pi e). \tag{14}$$

The equality takes place for squeezed states with the wave function

$$\psi(x) = (2\pi\sigma_{x^2})^{-1/4} \, e^{-x^2/4\sigma_{x^2}}. \tag{15}$$

Thus, one has

$$S_x = \frac{1}{2} \ln(2\pi e \sigma_{x^2}), \qquad S_p = \frac{1}{2} \ln(2\pi e \sigma_{p^2}), \tag{16}$$

where σ_{x^2} and σ_{p^2} read

$$\sigma_{x^2} = \int x^2 |\psi(x)|^2 \, dx - \left(\int x |\psi(x)|^2 \, dx \right)^2,$$
$$\sigma_{p^2} = \int p^2 |\widetilde{\psi}(p)|^2 \, dp - \left(\int p |\widetilde{\psi}(p)|^2 \, dp \right)^2, \tag{17}$$

and

$$\sigma_{x^2} \sigma_{p^2} = \frac{1}{4}. \tag{18}$$

For squeezed and correlated [5] states, the wave functions have the Gaussian form, i.e.,

$$\psi(x) = \mathcal{N} \exp(-ax^2 + bx), \qquad a = a_1 + ia_2,$$

and

$$\sigma_{x^2} \sigma_{p^2} = \frac{1}{4} \frac{1}{1 - R^2}. \tag{19}$$

Here R is the correlation coefficient of the position and momentum, i.e.,

$$R = \frac{\frac{1}{2}\langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle}{\sqrt{\sigma_{x^2} \sigma_{p^2}}}, \qquad |R| < 1, \tag{20}$$

and for squeezed but not correlated states $R = 0$.

The sum of entropies for the squeezed and correlated states reads

$$S_x + S_p = \ln(\pi e) + \ln \frac{1}{\sqrt{1 - R^2}} \geq \ln(\pi e). \tag{21}$$

For squeezed but not correlated states, the entropy S_x differs from S_p.

For multimode systems (multicomponent signals), the entropy uncertainty relation reads

$$S_{\vec{x}} + S_{\vec{p}} \geq N \ln(\pi e), \tag{22}$$

where N is the number of degrees of freedom of the system and

$$S_{\vec{x}} = -\int |\psi(\vec{x})|^2 \ln |\psi(\vec{x})|^2 \, d\vec{x},$$
$$S_{\vec{p}} = -\int |\widetilde{\psi}(\vec{p})|^2 \ln |\widetilde{\psi}(\vec{p})|^2 \, d\vec{p}. \tag{23}$$

The functions $\psi(\vec{x})$ and $\widetilde{\psi}(\vec{p})$ are connected by the Fourier transform

$$\widetilde{\psi}(\vec{p}) = (2\pi)^{-N/2} \int \psi(\vec{x}) e^{-i\vec{p}\vec{x}} \, d\vec{x}. \tag{24}$$

For the Gaussian wave function corresponding to factorized squeezed state of several modes,

$$S_{\vec{x}} + S_{\vec{p}} = N \ln(\pi e). \tag{25}$$

3 Symplectic tomography

There exists an invertable map of the density operator (matrix) $\hat{\rho}$ onto the symplectic tomogram [25,26] which is the probability-density of random quadrature X

$$w(X, \mu, \nu) = \mathrm{Tr}\, \hat{\rho} \; \delta(X - \mu\hat{q} - \nu\hat{p}). \tag{26}$$

Here $\hat{\rho}$ is the density operator. The parameters μ and ν are real parameters and operators $\hat{q}$ and $\hat{p}$ are quadrature operators.

The map (26) has the inverse [17]

$$\hat{\rho} = \frac{1}{2\pi} \int w(X, \mu, \nu) \exp\left[i \left(X - \mu\hat{q} - \nu\hat{p}\right)\right] dX \, d\mu \, d\nu. \tag{27}$$

For a pure state $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$, the transform (26) yields [27]

$$w(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \psi(y) \exp\left(\frac{i\mu}{2\nu} y^2 - \frac{iX}{\nu} y\right) dy \right|^2. \tag{28}$$

The function $w(X, \mu, \nu)$ (the state tomogram) is the probability density of the position X, i.e.,

$$w(X, \mu, \nu) \geq 0$$

and

$$\int w(X, \mu, \nu) \, dX = 1. \tag{29}$$

Transformation (26) can be expressed in terms of the real Wigner function [28]

$$W(q, p) = \int \rho\left(q + \frac{u}{2}, q - \frac{u}{2}\right) e^{-ipu} \, du, \tag{30}$$

where $\rho(x, x')$ is the density matrix in the position representation, and for $\mathrm{Tr}\, \hat{\rho} = 1$ one has

$$\int W(q, p) \, \frac{dq \, dp}{2\pi} = 1. \tag{31}$$

The density matrix reads

$$\rho(x, x') = \frac{1}{2\pi} \int W\left(\frac{x + x'}{2}, p\right) e^{ip(x - x')} \, dp. \tag{32}$$

In terms of the tomogram, one has

$$\rho(x, x') = \frac{1}{2\pi} \int w(Y, \mu, x - x') e^{-i[Y - \mu(x + x')/2]} \, dY \, d\mu. \tag{33}$$

The tomographic-probability density has the homogeneity property following from its definition (26) and the relation for the Dirac delta-function

$$\delta(\lambda y) = \frac{1}{|\lambda|} \delta(y). \tag{34}$$

The homogeneity property reads

$$w(\lambda X, \lambda\mu, \lambda\nu) = \frac{1}{|\lambda|} w(X, \mu, \nu). \tag{35}$$

Also for the pure state, one has

$$w(X, 1, 0) = |\psi(X)|^2 \tag{36}$$

and

$$w(X, 0, 1) = |\widetilde{\psi}(X)|^2, \tag{37}$$

where $\psi(X)$ is the wave function in the position representation and $\widetilde{\psi}(X)$ is the wave function in the momentum representation.

These properties are connected with the expression of symplectic tomogram in terms of the Wigner function [25]

$$w(X, \mu, \nu) = \int W(q, p) \, \delta(X - \mu q - \nu p) \, \frac{dq \, dp}{2\pi}. \tag{38}$$

The inverse transform reads

$$W(q, p) = \frac{1}{2\pi} \int w(X, \mu, \nu) \exp\left[i \left(X - \mu q - \nu p\right)\right] \, dX \, d\mu \, d\nu. \tag{39}$$

Since for the pure state,

$$\int W(q, p) \frac{dp}{2\pi} = |\psi(q)|^2 \tag{40}$$

and

$$\int W(q, p) \frac{dq}{2\pi} = |\widetilde{\psi}(p)|^2, \tag{41}$$

relations (36) and (37) are easily obtained.

Tomogram (28) can be rewritten in the form

$$w(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \psi(y) \exp\left[\frac{i}{2}\left(\frac{\mu}{\nu} y^2 - \frac{2X}{\nu} y + \frac{\mu}{\nu} X^2\right)\right] dy \right|^2. \tag{42}$$

For $\mu = \cos t$ and $\nu = \sin t$, one has the optical tomogram [29,30]

$$w(X,t) = \left| \int \psi(y) \exp\left[\frac{i}{2} \left(\cot t \ (y^2 + X^2) - \frac{2X}{\sin t}\, y \right) \right] \frac{dy}{\sqrt{2\pi i \sin t}} \right|^2 . \quad (43)$$

On the other hand, this tomogram formally equals to

$$w(X,t) = |\psi(X,t)|^2, \quad (44)$$

where the wave function reads

$$\psi(X,t) = \frac{1}{\sqrt{2\pi i \sin t}} \int \exp\left[\frac{i}{2} \left(\cot t \ (y^2 + X^2) - \frac{2X}{\sin t}\, y \right) \right] \psi(y)\, dy, \quad (45)$$

being the fractional Fourier transform of the wave function $\psi(y)$. This wave function corresponds to the wave function of a harmonic oscillator with $\hbar = m = \omega = 1$ taken at the time moment t provided the wave function at the initial time moment $t = 0$ equals to $\psi(y)$.

For mixed state with density operator $\hat{\rho}$ written in the form of the spectral decomposition,

$$\hat{\rho} = \sum_k \lambda_k \mid \psi_k \rangle \langle \psi_k \mid, \quad (46)$$

where λ_k are nonnegative eigenvalues and $\mid \psi_k \rangle$ are the eigenvectors of the density operator, the optical tomogram reads

$$w(X, \mu = \cos t, \nu = \sin t) = \sum_k \frac{\lambda_k}{2\pi |\sin t|} \left| \psi_k(y) \times \exp\left[\frac{i}{2} \left(\cot t \ (y^2 + X^2) - \frac{2X}{\sin t} \right) \right] dy \right|^2 . \quad (47)$$

In equations (45) and (47), we use the identity of the kernel of fractional Fourier transform to the Green function of the Schrödinger evolution equation for the harmonic oscillator [31].

Tomogram of a mixed state takes the form of convex sum of tomograms of pure states $\mid \psi_k \rangle$, i.e.,

$$w(X,\mu,\nu) = \sum_k \lambda_k w_k(X,\mu,\nu), \quad (48)$$

where $w_k(X,\mu,\nu)$ are given by equation (42) with

$$\psi(y) \to \psi_k(y) = \langle y \mid \psi_k \rangle.$$

In view of equations (36) and (37), one has for mixed state

$$w(X,1,0) = \sum_k \lambda_k |\psi_k(X)|^2 \quad (49)$$

and

$$w(X,0,1) = \sum_k \lambda_k |\widetilde{\psi}_k(X)|^2, \quad (50)$$

where $\psi_k(X)$ is the complex wave function in the position representation of the eigenstate $\mid \psi_k \rangle$ and $\widetilde{\psi}_k(X)$ is the complex wave function in the momentum representation of this state.

Thus, we pointed out that symplectic tomogram of a quantum state can be interpreted as modulus squared of the harmonic-oscillator's wave function for pure state or as convex sum of modulus squared of such functions for mixed state.

Another possibility for analogous interpretation follows in view of considering the tomogram $w(X,\mu,\nu)$ within the framework of Fresnel-tomography approach [32].

In fact, formula (28) with $\mu = 1$ can be rewritten in the form

$$w_{\rm F}(X,\nu) \equiv w(X,\mu=1,\nu) = \left| \int \frac{1}{\sqrt{2\pi i \nu}} \exp\left[\frac{i(X-y)^2}{2\nu} \right] \psi(y)\, dy \right|^2 . \quad (51)$$

The Fresnel tomogram $w_{\rm F}(X,\nu)$ is related to the optical tomogram by

$$w_{\rm F}\left(\frac{X}{\mu}, \frac{\nu}{\mu} \right) = |\mu| w(X,\mu,\nu). \quad (52)$$

On the other hand, $w_{\rm F}(X/\mu, \nu/\mu)$ can be considered as the wave function of free particle at the time moment $t = \nu$ if the initial value of the wave function at the time moment $t = 0$ is equal to $\psi(y)$.

Thus, the Fresnel tomogram for pure state can be interpreted as modulus squared of the wave function of free particle.

The Fresnel tomogram is the probability distribution satisfying the normalization condition

$$\int w_{\rm F}(X,\nu)\, dX = 1. \quad (53)$$

For mixed state (46), the Fresnel tomogram reads

$$w_{\rm F}(X,\nu) = \sum_k \lambda_k \left| \frac{1}{\sqrt{2\pi i \nu}} \int \exp\left[\frac{i(X-y)^2}{2\nu} \right] \psi_k(y)\, dy \right|^2 \quad (54)$$

and

$$w_{\rm F}(X,0) = \sum_k \lambda_k |\psi_k(X)|^2. \quad (55)$$

4 Tomographic entropies

Since the symplectic tomogram has the standard probability distribution features, one can introduce entropy associated with the tomogram of quantum state [18] or of

analytic signal [19]. Thus one has entropy as the function of two real variables

$$S(\mu,\nu) = -\int w(X,\mu,\nu)\,\ln w(X,\mu,\nu)\,dX. \tag{56}$$

In view of the homogeneity and normalization conditions for tomogram (35), (29) one has the additivity property

$$S(\lambda\mu,\lambda\nu) = S(\mu,\nu) + \ln|\lambda|. \tag{57}$$

For pure state $|\,\psi\rangle$, one obtains the entropies S_x and S_p, namely,

$$S(1,0) = S_x \tag{58}$$

and

$$S(0,1) = S_p. \tag{59}$$

In view of inequality (10), one has the inequality for tomographic entropies

$$S(1,0) + S(0,1) \geq \ln(\pi e). \tag{60}$$

For multimode system, the symplectic entropy reads

$$S(\vec{\mu},\vec{\nu}) = -\int w(\vec{X},\vec{\mu},\vec{\nu})\,\ln w(\vec{X},\vec{\mu},\vec{\nu})\,d\vec{X}. \tag{61}$$

Since the symplectic entropy is related to entropies $S_{\vec{x}}$ and $S_{\vec{p}}$ of the multimode-system state, one can use inequality (22) to obtain the entropic uncertainty relation in the form of inequality for symplectic entropies

$$S(\vec{1},\vec{0}) + S(\vec{0},\vec{1}) \geq N\ln(\pi e), \tag{62}$$

where $\vec{\mu} = \vec{1}$ means $\vec{\mu} = (1,1,\ldots,1)$ and $\vec{\nu} = \vec{1}$ means $\vec{\nu} = (1,1,\ldots,1)$.

The Fresnel tomogram provides the Fresnel entropy of the quantum state

$$S_{\mathrm{F}}(\nu) = -\int w_{\mathrm{F}}(X,\nu)\,\ln w_{\mathrm{F}}(X,\nu)\,dX. \tag{63}$$

It can be readily seen that the Fresnel entropy $S_{\mathrm{F}}(\nu)$ can be easily obtained from the symplectic entropy (56) choosing $\mu = 1$, i.e.,

$$S(1,\nu) = S_{\mathrm{F}}(\nu). \tag{64}$$

This also means that

$$S_{\mathrm{F}}(0) = S_x. \tag{65}$$

For the optical tomogram (43), entropy is defined by the formula

$$S(t) = -\int w(X,t)\,\ln w(X,t)\,dX. \tag{66}$$

For the pure state, one has

$$S(0) = S_x \tag{67}$$

and

$$S(\pi/2) = S_p. \tag{68}$$

In view of the expression of tomogram in terms of wave function (44) and (45), one has the entropic uncertainty relation in the form

$$S(t) + S(t+\pi/2) \geq \ln \pi e. \tag{69}$$

Since symplectic and optical tomograms are connected as follows:

$$w(X,\mu=\cos t,\nu=\sin t) = w(X,t), \tag{70}$$

the corresponding entropies are also connected

$$S(t) = S(\mu=\cos t,\nu=\sin t). \tag{71}$$

For given symplectic entropy of any pure state $S(\mu,\nu)$, inequality (69) reads

$$S(\cos t,\sin t) + S(-\sin t,\cos t) \geq \ln \pi e. \tag{72}$$

The optical tomogram $w(x,t)$ and symplectic tomogram $w(X,\mu,\nu)$ connected by (70) can be related by another formula

$$w(X,\mu,\nu) = \frac{1}{\sqrt{\mu^2+\nu^2}}\,w\left(\frac{X}{\sqrt{\mu^2+\nu^2}},t\right). \tag{73}$$

This means that for given optical tomogram $w(x,t)$ one can reconstruct symplectic tomogram $w(X,\mu,\nu)$. Inserting equation (73) into basic equation defining the entropy (56) yields the equality

$$S(t) = S\Big(\sqrt{\mu^2+\nu^2}\cos t,\sqrt{\mu^2+\nu^2}\sin t\Big) - \frac{1}{2}\,\ln(\mu^2+\nu^2). \tag{74}$$

For symplectic entropies (72), the entropic uncertainty relation yields

$$\begin{aligned} &S\Big(\sqrt{\mu^2+\nu^2}\cos t,\sqrt{\mu^2+\nu^2}\sin t\Big)\\ &+S\Big(-\sqrt{\mu^2+\nu^2}\sin t,\sqrt{\mu^2+\nu^2}\cos t\Big)\\ &-\ln(\mu^2+\nu^2) \geq \ln \pi e. \end{aligned} \tag{75}$$

The extension of this inequality for multimode system reads

$$\begin{aligned} &S\Big(\sqrt{\mu_1^2+\nu_1^2}\cos t_1,\sqrt{\mu_2^2+\nu_2^2}\cos t_2,\ldots,\sqrt{\mu_N^2+\nu_N^2}\cos t_N,\\ &\sqrt{\mu_1^2+\nu_1^2}\sin t_1,\sqrt{\mu_2^2+\nu_2^2}\sin t_2,\ldots,\sqrt{\mu_N^2+\nu_N^2}\sin t_N\Big)\\ &+S\Big(-\sqrt{\mu_1^2+\nu_1^2}\sin t_1,-\sqrt{\mu_2^2+\nu_2^2}\sin t_2,\ldots,\\ &-\sqrt{\mu_N^2+\nu_N^2}\sin t_N,\\ &\sqrt{\mu_1^2+\nu_1^2}\cos t_1,\sqrt{\mu_2^2+\nu_2^2}\cos t_2,\ldots,\sqrt{\mu_N^2+\nu_N^2}\cos t_N\Big)\\ &-\sum_{k=1}^{N}\ln\left(\mu_k^2+\nu_k^2\right) \geq N\ln(\pi e), \end{aligned} \tag{76}$$

where entropy $S(\vec{\mu},\vec{\nu})$ is given by (61).

Tomogram of the ground state of multimode isotropic harmonic oscillator with unit masses and frequencies has the form

$$w_0\left(\vec{X},\vec{\mu},\vec{\nu}\right)=\prod_{k=1}^{N}\frac{1}{\sqrt{\pi\left(\mu_k^2+\nu_k^2\right)}}\,\exp\left(-\frac{X_k^2}{\mu_k^2+\nu_k^2}\right). \tag{77}$$

Entropy associated with this tomogram reads

$$S_0\left(\vec{\mu},\vec{\nu}\right)=\frac{N}{2}\,\ln\pi+\frac{N}{2}+\frac{N}{2}\sum_{k=1}^{N}\ln\left(\mu_k^2+\nu_k^2\right). \tag{78}$$

This entropy does not depend on the parameter t_k.

One can check that, if $\mu_k\to\sqrt{\mu_k^2+\nu_k^2}\cos t_k$ and $\nu_k\to\sqrt{\mu_k^2+\nu_k^2}\sin t_k$ in formula (78), relation (76) yields for S_0 the equality

$$\begin{aligned}
&S_0\Big(\sqrt{\mu_1^2+\nu_1^2}\cos t_1,\sqrt{\mu_2^2+\nu_2^2}\cos t_2,\ldots,\sqrt{\mu_N^2+\nu_N^2}\cos t_N,\\
&\sqrt{\mu_1^2+\nu_1^2}\sin t_1,\sqrt{\mu_2^2+\nu_2^2}\sin t_2,\ldots,\sqrt{\mu_N^2+\nu_N^2}\sin t_N\Big)\\
&+S\Big(-\sqrt{\mu_1^2+\nu_1^2}\sin t_1,-\sqrt{\mu_2^2+\nu_2^2}\sin t_2,\ldots,\\
&-\sqrt{\mu_N^2+\nu_N^2}\sin t_N,\\
&\sqrt{\mu_1^2+\nu_1^2}\cos t_1,\sqrt{\mu_2^2+\nu_2^2}\cos t_2,\ldots,\sqrt{\mu_N^2+\nu_N^2}\cos t_N\Big)\\
&-\sum_{k=1}^{N}\ln\left(\mu_k^2+\nu_k^2\right)=N\ln(\pi e).
\end{aligned} \tag{79}$$

5 Entropic inequality for solitons

Entropy of the soliton solution to nonlinear equations was discussed in [20]. In particular, the soliton solution to Gross-Pitaevskii equation [33] was considered in the tomographic-probability representation to study Bose-Einstein condensate (BEC) (see also [34,35]).

BEC soliton under consideration is given as the function

$$\psi(x)=\frac{1}{\sqrt{2l_z}}\,\mathrm{sech}\left(\frac{x}{l_z}\right), \tag{80}$$

where the parameter l_z describes the soliton width. Symplectic tomogram of BEC soliton reads

$$w_{\mathrm{S}}(X,\mu,\nu)=\frac{1}{2\pi|\nu|}\left|\int\frac{1}{\sqrt{2l_z}}\,\mathrm{sech}\left(\frac{y}{l_z}\right)\times\exp\left(\frac{i\mu}{2\nu}\,y^2-\frac{iX}{\nu}\,y\right)dy\right|^2, \tag{81}$$

where $\mu=r\cos t$ and $\nu=r\sin t$.

Since $\int|\psi(x)|^2dx=1$, tomogram (81) is nonnegative normalized probability distribution of random position X.

The tomographic entropy of BEC soliton equals to

$$\begin{aligned}
S(r,t)=-\int\frac{1}{2\pi|\nu|}\left|\int\frac{1}{\sqrt{2l_z}}\,\mathrm{sech}\left(\frac{y}{l_z}\right)\right.\\
\left.\times\exp\left(\frac{i\mu}{2\nu}\,y^2-\frac{iX}{\nu}\,y\right)dy\right|^2\\
\times\ln\left\{\frac{1}{2\pi|\nu|}\left|\int\frac{1}{\sqrt{2l_z}}\,\mathrm{sech}\left(\frac{y}{l_z}\right)\right.\right.\\
\left.\left.\exp\left(\frac{i\mu}{2\nu}\,y^2-\frac{iX}{\nu}\,y\right)dy\right|^2\right\}dX.
\end{aligned} \tag{82}$$

We introduce the function

$$F(r,t)=S(r,t)+S(r,t+\pi/2)-\ln r^2-\ln(\pi e). \tag{83}$$

According to the entropic uncertainty relation (75) this function (we call it entropic uncertainty function) must be nonnegative. Equation (74) and the additivity property (57) mean that the entropic uncertainty function (83) does not depend on parameter r.

Plots of function (83) for the Gaussian state and for the soliton are presented below.

(i) The normalized initial Gaussian profile is given by

$$\mathcal{F}_{\mathrm{G}}(y)=\frac{\exp(-y^2/2\sigma^2)}{\pi^{1/4}\sigma^{1/2}}, \tag{84}$$

where σ is the waist of Gaussian profile. The corresponding tomogram calculated with the help of equation (28) with $\mu=r\cos t$ and $\nu=r\sin t$ is given by

$$w_{\mathrm{G}}(r,t)=\frac{\sigma}{r\sqrt{\pi(\sin^2 t+\sigma^4\cos^2 t)}}\times\exp\left[-\frac{\sigma^2X^2}{r^2(\sin^2 t+\sigma^4\cos^2 t)}\right]. \tag{85}$$

The symplectic Gaussian entropy is given as follows:

$$S_{\mathrm{G}}(r,t)=\frac{1}{2}-\ln\left[-\frac{\sigma}{r\sqrt{\pi(\sin^2 t+\sigma^4\cos^2 t)}}\right]. \tag{86}$$

The corresponding entropic uncertainty function $F_{\mathrm{G}}(t)$ in this case can be calculated explicitly

$$F_{\mathrm{G}}(t)=\ln\left[\sqrt{1+\left(\frac{1-\sigma^4}{2\sigma^2}\right)^2\sin^2 2t}\right]. \tag{87}$$

Note that the positive definite function $F_{\mathrm{G}}(t)$ does not depend on the radial variable r, i.e., it is the same for both the symplectic and optical entropies and it reduces to zero for $\sigma=1$, whereas for $\sigma\neq1$ it is periodic with period $\pi/2$ as can be seen from Figure 1.

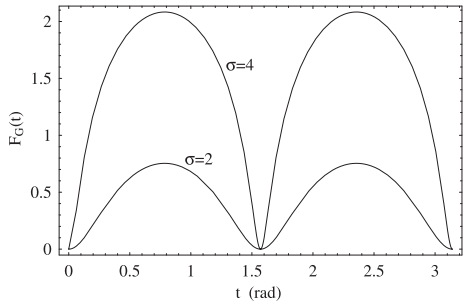

Fig. 1. Plot of function $F_{\rm G}(t)$ for two values of the Gaussian waist $\sigma = 2$ and $\sigma = 4$.

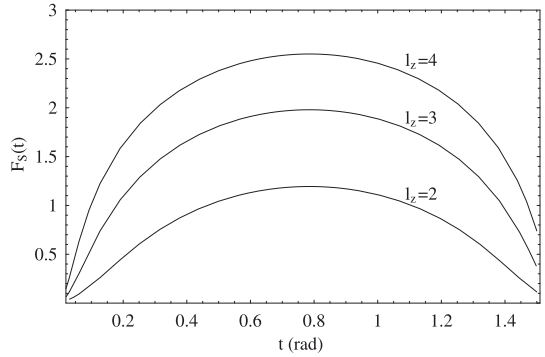

Fig. 2. Plot of entropic uncertainty function $F_{\rm S}(r,t)$ for three values of the soliton-width parameter $l_z = 2$, $l_z = 3$, and $l_z = 4$.

(ii) The initial profile of soliton is given by (80). Using the tomographic entropy (82), one can calculate the entropic uncertainty function (83) for the soliton numerically. The numerical result was obtained by speeding up the calculation procedure employing fast Fourier transform (FFT) method. Indeed, it was shown (see [32]) that tomogram can be expressed as convolution of the initial profile with a chirp function (CF). Here the convolution was computed via FFT, namely, the inverse Fourier transform of the product of FFT of the initial profile and FFT of CF. Plots in Figure 2 demonstrate the behaviour of entropic uncertainty function $F_{\rm S}(t)$ for different values of the soliton-width parameter l_z. One can see that the upper bound of this function depends on the soliton width.

Before concluding this section, we consider other examples of quantum states with generic Gaussian Wigner function and the corresponding tomogram

$$w(X,\mu,\nu) = \frac{1}{\sqrt{2\pi\sigma_{XX}(\mu,\nu)}} \exp\left(-\frac{X^2}{2\sigma_{XX}(\mu,\nu)}\right), \quad (88)$$

where

$$\sigma_{XX}(\mu,\nu) = \mu^2\sigma_{qq} + \nu^2\sigma_{pp} + 2\mu\nu\sigma_{qp}. \quad (89)$$

The parameters σ_{qq}, σ_{pp}, and σ_{qp} satisfy the uncertainty relation

$$\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq 1/4. \quad (90)$$

The state under consideration for

$$\sigma_{qq} = \sigma_{pp} = \frac{1}{2}\coth\frac{\beta}{2}, \quad \sigma_{qp} = 0 \quad (91)$$

is the oscillator quantum thermal state with temperature $T = \beta^{-1}$.

For the state (88), the entropic uncertainty function reads

$$S(t) = \ln 2 + \frac{1}{2}\ln\left[\sigma_{qq}\cos^2 t + \sigma_{pp}\sin^2 t + 2\sigma_{qp}\sin t\cos t\right] + \frac{1}{2}\ln\left[\sigma_{qq}\sin^2 t + \sigma_{pp}\cos^2 t - 2\sigma_{qp}\sin t\cos t\right]. \quad (92)$$

For squeezed thermal state we have

$$\sigma_{qq} = \frac{\lambda}{2}\coth\frac{1}{2\beta}, \quad \sigma_{pp} = \frac{1}{2\lambda}\coth\frac{1}{2\beta}, \quad \sigma_{qp} = 0, \quad (93)$$

where λ is squeezing parameter.

6 Conclusions

To conclude, we point out the main results of this paper.

Inequalities (69) and (75) being the generalizations of known entropic inequalities for probability distributions of conjugate position and momentum are obtained for entropies associated with symplectic tomograms.

The new uncertainty relations obtained characterize the behavior of quantum state in quantum mechanics as well as the behavior of analytic signal in signal analysis. The entropic uncertainty relation for tomographic entropy are obtained also for multimode quantum state. The uncertainty relation is given by formula (76). The entropy under study as any Shannon entropy provides the informational characteristics of the signal.

The nonnegative entropic uncertainty function introduced can be used to characterize the Shannon information content of a signal, e.g., of optical signal.

The uncertainty relation for tomographic entropies is a new additional property of nonlinear signals including BEC solitons obeying the Gross-Pitaevskii equation. The physical meaning of tomographic entropic uncertainty relations will be deepen in a future work.

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